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# Growth properties of Nevanlinna matrices for rational moment problems

A. Bultheel[a,*], P. González-Vera[b], E. Hendriksen[c], O. Njåstad[d]

[a] *Department of Computer Science, K.U.Leuven, Celestijnenlaan 200A, 3001 Heverlee, Belgium*
[b] *Department of Mathematical Analysis, University of La Laguna, Tenerife, Spain*
[c] *Beerstratenlaan 23, 2421GN Nieuwkoop, The Netherlands*
[d] *Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway*

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**Abstract**

We consider rational moment problems on the real line with their associated orthogonal rational functions. There exists a Nevanlinna-type parameterization relating to the problem, with associated Nevanlinna matrices of functions having singularities in the closure of the set of poles of the rational functions belonging to the problem. We prove results related to the growth at the singularities of the functions in a Nevanlinna matrix, and in particular provide bounds on the growth analogous to the corresponding result in the classical polynomial case, when the number of singularities is finite.


*Keywords:* Rational moment problems; Orthogonal rational functions; Nevanlinna parameterization; Riesz-type criterion; Growth estimates

## 1. Introduction

We use the following notation. $\mathbb{C}$ denotes the complex plane, $\hat{\mathbb{C}}$ the one-point compactification of $\mathbb{C}$ (the extended complex plane), $\mathbb{R}$ the real line, $\hat{\mathbb{R}}$ the closure of $\mathbb{R}$ in $\hat{\mathbb{C}}$, $\mathbb{U}$ the open upper half-plane, and $\hat{\mathbb{U}}$ the closure of $\mathbb{U}$ in $\hat{\mathbb{C}}$.

* Corresponding author.
*E-mail addresses:* adhemar.bultheel@cs.kuleuven.be (A. Bultheel), pglez@ull.es (P. González-Vera), erik.hendriksen@planet.nl (E. Hendriksen), njastad@math.ntnu.no (O. Njåstad).

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A function $f$ is called a *Pick function* if it is holomorphic in $\mathbb{U}$ and maps $\mathbb{U}$ into $\hat{\mathbb{U}}$. A Pick function is either a constant in $\hat{\mathbb{R}}$ or maps $\mathbb{U}$ into $\mathbb{U}$.

Let $\mu$ be a finite positive measure on $\mathbb{R}$. The *Stieltjes transform* $S_\mu$ of $\mu$ is defined as

$$S_\mu(z) = \int_{\mathbb{R}} C(t, z) \mathrm{d}\mu(t), \quad C(t, z) = \frac{1}{t - z}.$$

The *Herglotz–Riesz–Nevanlinna transform* $\Omega_\mu$ of $\mu$ is defined as

$$\Omega_\mu(z) = \int_{\mathbb{R}} D(t, z) \mathrm{d}\mu(t), \quad D(t, z) = \frac{1 + tz}{t - z}.$$

Both of these functions are Pick functions. Furthermore,

$$\Omega_\mu(z) = (1 + z^2) S_\mu(z) + z \int_{\mathbb{R}} \mathrm{d}\mu(t).$$

Thus for fixed $z$ there is a one-to-one correspondence between $\Omega_\mu$ and $S_\mu$ as functions of $\mu$.

Let $M$ be a Hermitian, positive definite linear functional on the space $\mathcal{P}$ of polynomials, and define its moments $c_n$ by $c_n = M[z^n]$, $n = 0, 1, 2, \ldots$. A solution of the *Hamburger moment problem* for $\{c_n\}$ (or $M$) is a positive measure $\mu$ on $\mathbb{R}$ which satisfies $\int_{\mathbb{R}} t^n \mathrm{d}\mu(t) = c_n$ for all $n$. (Such measures exist.) A moment problem is called *determinate* if it has exactly one solution, and *indeterminate* if it has more than one solutions.

There is a one-to-one correspondence between all Pick functions $f$ and all solutions $\mu$ of an indeterminate problem given by

$$S_\mu(z) = -\frac{A(z) f(z) - C(z)}{B(z) f(z) - D(z)}.$$

(*Nevanlinna parameterization* of the solutions). Here $A, B, C, D$ are entire transcendent functions where the growth is restricted as follows. Let $F$ be any of the functions $A$, $B$, $C$, $D$. Then, for every positive $\varepsilon$, there exists a constant $M(\varepsilon)$ such that

$$|F(z)| \leq M(\varepsilon) \exp\{\varepsilon |z|\}.$$

(Thus the function is of at most minimal type of order 1.)

For detailed treatments of important aspects of the Hamburger moment problem, see, e.g. [1,3–5,11–13,17,23–27].

The *strong Hamburger moment problem* is analogous to the classical problem, with the space of polynomials replaced by the space of Laurent polynomials (linear combinations of $z^k$, $k = 0, \pm 1, \pm 2, \ldots$). A similar parameterization of the set of solutions of an indeterminate problem holds, with the appropriate functions $A$, $B$, $C$, $D$ holomorphic in $\mathbb{C} \setminus \{0\}$. When $F$ is any of the functions $A$, $B$, $C$, $D$, there exist, for every positive $\varepsilon$, constants $M_\infty(\varepsilon)$ and $M_0(\varepsilon)$ such that

$$|F(z)| \leq M_\infty(\varepsilon) \exp(\varepsilon |z|) \quad \text{and} \quad |F(z)| \leq M_0(\varepsilon) \exp(\varepsilon / |z|).$$

For detailed treatments on the theory of strong Hamburger moment problems, see, e.g., [14,18–22].

In this paper, we treat a *rational moment problem*, where polynomials are replaced by rational functions with prescribed poles in $\hat{\mathbb{R}}$. A Nevanlinna parameterization for solutions of an indeterminate problem in terms of $\Omega_\mu$ and Pick functions was proved by Almendral in [2, Theorem 9].

The classical Hamburger moment problem is a special case of the rational problem under consideration (since polynomials are rational functions with all their poles at infinity). Thus in this case there is an alternative parameterization in terms of $\Omega_\mu$ as well (as opposed to the parameterization in terms of $S_\mu$ as described above).

Our aim in this paper is to establish growth conditions at the singularities of the functions $A, B, C, D$ appearing in the parameterization formula.

In Section 2, we introduce the rational spaces on which the rational moment problems are defined, and sketch the theory of orthogonal rational functions and their use in the theory of rational moment problems, including the Nevanlinna parameterization of the solutions of indeterminate problems. Section 3 is devoted to establishing a Riesz-type criterion for such indeterminate problems when the number of singularities is finite. This criterion is crucial for the further development of the growth properties. (For the classical Riesz criterion, see, e.g., [1,23–25].) Finally, in Section 4, we prove our result on the restriction on the growth of the functions $A, B, C, D$ at the singularities.

The organization and presentation of the material in Sections 3 and 4 is strongly influenced by Akhiezer's work [1] on the classical moment problem. Other very instructive treatments of the classical problem can be found in the treatises by Riesz [23–25] and by Shohat and Tamarkin [26] and Stone [27]. This classical approach has to be modified in a number of ways, but the final results are of basically the same structure.

**Remark 1.1.** A parameterization result for rational moment problems associated with poles outside the closed unit disk and measures on the unit circle $\mathbb{T}$ was proved in [10]. Here $\Omega_\mu$ is replaced by the Herglotz–Riesz transform $\int_{\mathbb{T}} \frac{t+z}{t-z} \mathrm{d}\mu(t)$, and Pick functions are replaced by Carathéodory functions (holomorphic in the open unit disk and mapping this disk to the closed right half-plane). All the isolated singularities of the relevant functions are poles in this case.

## 2. Orthogonal rational functions and rational moment problems

Let $\{\alpha_k\}_{k=1}^\infty$ be a sequence of arbitrary points (interpolation points or singularities) in $\hat{\mathbb{R}} \setminus \{0\}, \alpha_0 = \infty$. We denote by $G$ the set of points $\alpha$ in $\hat{\mathbb{R}} \setminus \{0\}$ for which there is at least one $k$ such that $\alpha_k = \alpha$. For $\alpha \in G$, we denote by $\Gamma_\alpha$ the subsequence of $\{\alpha_k\}_{k=1}^\infty$ consisting of those $\alpha_k$ for which $\alpha_k = \alpha$.

Set

$$\pi_0 = 1, \qquad \pi_n(z) = \prod_{k=1}^n \left(1 - \frac{z}{\alpha_k}\right), \quad n = 1, 2, \ldots,$$

$$b_n(z) = \frac{z^n}{\pi_n(z)}, \quad n = 0, 1, 2, \ldots.$$

The set $\{b_0, b_1, \ldots, b_n\}$ is a basis for the space

$$\mathcal{L}_n = \left\{ \frac{p(z)}{\pi_n(z)} : p \in \mathcal{P}_n \right\},$$

where $\mathcal{P}_n$ denotes the space of polynomials of degree at most $n$. We set $\mathcal{L}_\infty = \cup_{n=0}^\infty \mathcal{L}_n$. We shall also consider the space $\mathcal{R}_\infty = \mathcal{L}_\infty \cdot \mathcal{L}_\infty$ consisting of products of two functions in $\mathcal{L}_\infty$. Note that, if $\Gamma_\alpha$ is infinite for all $\alpha \in G$, then $\mathcal{R}_\infty = \mathcal{L}_\infty$.

**Remark 2.1.** The space of Laurent polynomials is not formally included in this setting. The exclusion of the origin as an interpolation point is for technical reasons. A discussion of basic properties in the general case when the origin is also included among the possible interpolation points can be found in [9].

Let $M$ be a Hermitian, positive definite linear functional on $\mathcal{R}_\infty$. Thus $M[\overline{f}] = \overline{M[f]}$ for $f \in \mathcal{R}_\infty$ and $M[g \cdot \overline{g}] > 0$ for $g \in \mathcal{L}_\infty$, $g \neq 0$. For convenience, we assume $M$ to be normalized such that $M[1] = 1$. The moments $\mu_{m,n}$ of $M$ are defined as

$$\mu_{m,n} = M[b_m \cdot b_n].$$

(Note that $\overline{b}_n = b_n$.) A positive measure $\mu$ on $\mathbb{R}$ is said to solve the *rational moment problem for the functional $M$ on $\mathcal{L}_\infty$* if $b_m$ is integrable with respect to $\mu$ and

$$\int_{\mathbb{R}} b_m(t)\mathrm{d}\mu(t) = \mu_{m,0} \quad \text{for } m = 0, 1, 2, \ldots.$$

Equivalently,

$$\int_{\mathbb{R}} g(t)\mathrm{d}\mu(t) = M[g] \quad \text{for } g \in \mathcal{L}_\infty.$$

A measure $\mu$ on $\mathbb{R}$ is said to solve the *rational moment problem on $\mathcal{R}_\infty$* if $b_m \cdot b_n$ is integrable with respect to $\mu$ and

$$\int_{\mathbb{R}} b_m(t)b_n(t)\mathrm{d}\mu(t) = \mu_{m,n} \quad \text{for } m, n = 0, 1, 2, \ldots.$$

Equivalently,

$$\int_{\mathbb{R}} f(t)\mathrm{d}\mu(t) = M[f] \quad \text{for } f \in \mathcal{R}_\infty.$$

A solvable rational moment problem is said to be *determinate* if it has exactly one solution, and *indeterminate* if it has more than one solution. We denote by $\mathcal{M}(\mathcal{L}_\infty)$ the set of solutions of the problem on $\mathcal{L}_\infty$, and by $\mathcal{M}(\mathcal{R}_\infty)$ the set of solutions of the problem on $\mathcal{R}_\infty$.

Let $\{\varphi_n\}_{n=0}^\infty$ be the sequence of functions obtained by orthonormalization (with respect to $M$) of the sequence $\{b_n\}_{n=0}^\infty$. We fix them uniquely by multiplying with a unimodular constant, so that the coefficient of $b_n$ in the expansion of $\varphi_n$ with respect to the basis $\{b_n\}$ is positive.

The function $\varphi_n$ has the form $\varphi_n(z) = \frac{p_n(z)}{\pi_n(z)}$, $p_n \in \mathcal{P}_n$. Note that, by our normalization, the coefficients are real; hence $\varphi_n(x)$ is real for $x \in \mathbb{R}$. The functions $\psi_n$ of the second kind are defined by

$$\psi_0(z) = -z, \qquad \psi_n(z) = M_t\left[D(t,z)\{\varphi_n(t) - \varphi_n(z)\}\right], \quad n = 1, 2, \ldots,$$

where $M_t$ means that the functional $M$ operates on a function of $t$. We shall also consider the rational functions $\sigma_n$ given by ($M_t$ refers to $M$ applied to the $t$-variable)

$$\sigma_n(z) = M_t\left[C(t,z)\{\varphi_n(t) - \varphi_n(z)\}\right], \quad n = 0, 1, 2, \ldots.$$

We observe that both $\psi_n$ and $\varphi_n$ belong to $\mathcal{L}_n$, and that both functions are real for real $z$. Furthermore, we find that

$$\sigma_n(z) = \frac{1}{1+z^2}[z\varphi_n(z) + \psi_n(z)], \quad n = 0, 1, 2, \ldots. \tag{2.1}$$

The sequences $\{\varphi_n\}$, $\{\psi_n\}$, and $\{\sigma_n\}$ satisfy a three-term recurrence relation of the form

$$\begin{bmatrix}\sigma_n(z)\\ \psi_n(z)\\ \varphi_n(z)\end{bmatrix} = \left\{E_n \frac{z}{1-z/\alpha_n} + B_n \frac{1-z/\alpha_{n-2}}{1-z/\alpha_n}\right\} \begin{bmatrix}\sigma_{n-1}(z)\\ \psi_{n-1}(z)\\ \varphi_{n-1}(z)\end{bmatrix} + C_n \frac{1-z/\alpha_{n-2}}{1-z/\alpha_n} \begin{bmatrix}\sigma_{n-2}(z)\\ \psi_{n-2}(z)\\ \varphi_{n-2}(z)\end{bmatrix},$$

which holds for $n = 1, 2, \ldots$, if we start with the initial conditions

$$\left[\begin{array}{c|c}\sigma_{-1}(z) & \sigma_0(z)\\ \psi_{-1}(z) & \psi_0(z)\\ \varphi_{-1}(z) & \varphi_0(z)\end{array}\right] = \left[\begin{array}{c|c} 1 & 0\\ 1+z^2 & -z\\ 0 & 1\end{array}\right],$$

and provided we set by convention $\alpha_{-1} = \infty$ and $E_0 = -1$. Here $B_n, C_n, E_n$ are real numbers satisfying $E_n = -C_n E_{n-1}$ for $n = 1, 2, 3, \ldots$. See [8, Section 11.1, Theorems 11.1.2 and 11.2, Lemma 11.1.6].[1]

Note that $\sigma_1$ has the form $\sigma_1(z) = \kappa/(1 - z/\alpha_1)$, where $\kappa = C_1$ is a constant. We define

$$\chi_n(z) = \kappa^{-1}(1 - z/\alpha_1)\sigma_{n+1}(z), \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

Note that $\chi_n(x)$ is real for real $x$.

The sequence $\{\chi_n\}$ satisfies the recurrence relation

$$\chi_n(z) = \left\{E_{n+1} \frac{z}{1-z/\alpha_{n+1}} + B_{n+1} \frac{1-z/\alpha_{n-1}}{1-z/\alpha_{n+1}}\right\} \chi_{n-1}(z) + C_{n+1} \frac{1-z/\alpha_{n-1}}{1-z/\alpha_{n+1}} \chi_{n-2}(z)$$

for $n = 1, 2, 3, \ldots$, with $\chi_0 = 1$ and $\chi_{-1} = 0$.

Set

$$\tilde{\pi}_0 = 1, \qquad \tilde{\pi}_n(z) = \prod_{k=2}^{n+1}\left(1 - \frac{z}{\alpha_k}\right), \quad n = 1, 2, \ldots, \quad \text{and} \quad \tilde{b}_n(z) = \frac{z^n}{\tilde{\pi}_n(z)},$$

for $n = 0, 1, 2, \ldots$.

Let $\tilde{\mathcal{L}}_n$ denote the space spanned by $\{\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_n\}$, and set $\tilde{\mathcal{L}}_\infty = \cup_{n=0}^{\infty} \tilde{\mathcal{L}}_n$, $\tilde{\mathcal{R}}_\infty = \tilde{\mathcal{L}}_\infty \cdot \tilde{\mathcal{L}}_\infty$. We then have $\chi_n \in \tilde{\mathcal{L}}_n$.

According to the Favard-type theorem for orthogonal rational functions (see [8, Section 11.9, Theorem 11.9.4]), it follows that there is a positive functional $\tilde{M}$ on $\tilde{\mathcal{R}}_\infty$ such that the sequence $\{\chi_n\}$ is orthonormal with respect to $\tilde{M}$. We can then consider moment problems on $\tilde{\mathcal{L}}_\infty$ and $\tilde{\mathcal{R}}_\infty$ for the functional $\tilde{M}$. We shall call these moment problems *associated moment problems*. Since $\tilde{M}$ is positive, the moment problem on $\tilde{\mathcal{L}}_\infty$ is always solvable.

We shall use the notation

$$\omega_n(z) = 1 + \sum_{k=1}^{n-1} |\varphi_k(z)|^2, \qquad \Omega_n(z) = 1 + \sum_{k=1}^{n-1} |\psi_k(z)|^2, \qquad \tilde{\omega}_n(z) = 1 + \sum_{k=1}^{n-1} |\chi_k(z)|^2.$$

We also set

$$\omega_{\alpha,n}(z) = \sum_{\substack{k=1\\ \alpha_k \in \Gamma_\alpha}}^{n-1} |\varphi_k(z)|^2.$$

Note that $\omega_n(z) = 1 + \sum_{\alpha \in G} \omega_{\alpha,n}(z)$.

[1] Note that our definition of $D(t, z)$ and $C(t, z)$ differs slightly from the one given in [8], which gives a different normalization for $\psi_n$. Given the recursion for $\varphi_n$ and $\psi_n$, the recursion for $\sigma_n$ then follows from (2.1).

Let $x_0$ be a point in $\mathbb{R} \setminus [\hat{G} \cup \{0\}]$, where $\hat{G}$ denotes the closure in $\hat{\mathbb{C}}$ of the set $G$ of interpolation points. For technical reasons, $x_0$ is chosen such that $\psi_n(x_0) \neq 0$ and $q_n(\alpha_k, x_0) \neq 0$ for $k = 1, 2, \ldots, n$, for all $n$, where $q_n(z, \tau)$ is the numerator polynomial of the rational function $\varphi_n(z) + \tau \frac{1-z/\alpha_{n-1}}{1-z/\alpha_n} \varphi_{n-1}(z)$. Such a choice is always possible; see [8, Lemma 11.5.4]. In the following, $x_0$ shall be kept fixed, and will not be included in the notation for $A_n, B_n, C_n, D_n$ below. We set

$$H(z, x_0) = \frac{1}{z} - \frac{1}{x_0} = \frac{x_0 - z}{x_0 z}$$

and define

$$A_n(z) = H(z, x_0) \left[ 1 + \sum_{k=1}^{n-1} \psi_k(x_0)\psi_k(z) \right]$$

$$B_n(z) = H(z, x_0) \left[ D(z, x_0) - \sum_{k=1}^{n-1} \psi_k(x_0)\varphi_k(z) \right]$$

$$C_n(z) = H(z, x_0) \left[ D(z, x_0) + \sum_{k=1}^{n-1} \varphi_k(x_0)\psi_k(z) \right]$$

$$D_n(z) = H(z, x_0) \left[ 1 + \sum_{k=1}^{n-1} \varphi_k(x_0)\varphi_k(z) \right].$$

(Note that the definitions differ from those used in [2] by a real constant factor $E_n$.)

We set $\mathbb{C}_G = \hat{\mathbb{C}} \setminus [\hat{G} \cup \{-\mathrm{i}, \mathrm{i}\}]$. For $z \in \mathbb{C}_G$ and $t \in \hat{\mathbb{R}}$, we define

$$T_n(z, t) = -\frac{A_n(z)t - C_n(z)}{B_n(z)t - D_n(z)}$$

(which means $-A_n(z)/B_n(z)$ when $t = \infty$). The functions $A_n, B_n, C_n, D_n$ can also be expressed in the following way:

$$A_n(z) = \frac{1}{E_n x_0 z} \left[ f_n(x_0, z)\psi_n(x_0)\psi_{n-1}(z) - f_n(z, x_0)\psi_{n-1}(x_0)\psi_n(z) \right]$$

$$B_n(z) = \frac{1}{E_n x_0 z} \left[ f_n(x_0, z)\psi_n(x_0)\varphi_{n-1}(z) - f_n(z, x_0)\psi_{n-1}(x_0)\varphi_n(z) \right]$$

$$C_n(z) = \frac{1}{E_n x_0 z} \left[ f_n(x_0, z)\varphi_n(x_0)\psi_{n-1}(z) - f_n(z, x_0)\varphi_{n-1}(x_0)\psi_n(z) \right]$$

$$D_n(z) = \frac{1}{E_n x_0 z} \left[ f_n(x_0, z)\varphi_n(x_0)\varphi_{n-1}(z) - f_n(z, x_0)\varphi_{n-1}(x_0)\varphi_n(z) \right],$$

where $f_n(z, w) = \left(1 - \frac{z}{\alpha_{n-1}}\right)\left(1 - \frac{w}{\alpha_n}\right)$. See [2, Section 8].

It follows by a simple argument from [8, Corollary 11.5.6] that the functions $B_n(z)t - D_n(z)$, $t \in \hat{\mathbb{R}}$, have all their zeros on $\mathbb{R}$. According to [8, Lemma 11.10.6], the function $z \to T_n(z, t)$ for $t \in \hat{\mathbb{R}}$ is a Pick function; hence all the zeros of $A_n(z)t - C_n(z)$ are also real.

The index $n$ (or the function $\varphi_n$) is said to be *regular* if $p_n(\alpha_{n-1}) \neq 0$ ($p_n(\infty) \neq 0$ means that $p_n$ has degree exactly equal to $n$).

For $z$ fixed, the linear fractional transformation $t \to T_n(z, t)$ maps for a regular index $n$ the closed lower half-plane onto a proper closed disk $\Delta_n(z)$ in the open right half-plane. When

$m > n$, we have $\Delta_m(z) \subset \Delta_n(z)$. Let $\Lambda$ denote the sequence of regular indices, and set

$$\Delta_\infty(z) = \bigcap_{n\in\Lambda} \Delta_n(z).$$

Then $\Delta_\infty(z)$ is a proper, closed disk or a single point, independent of $z$ in $\mathbb{C}_G$. Furthermore, $\Delta_\infty(z)$ is a proper disk if and only if the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ converges locally uniformly in the domain $\mathbb{C}_G$. This is the case if and only if the series $\sum_{k=1}^{\infty} |\psi_k(z)|^2$ converges. This follows from the expression for the radius of the circle $\Delta_n$. See [8, Section 11.7, Theorem 11.7.1-5].

We shall in the following assume that the set $\Lambda$ *is infinite*. For simplicity of notation, we let without loss of generality $\Lambda$ consist of the natural numbers. We shall use the notation

$$\omega(z) = 1 + \sum_{k=1}^{\infty} |\varphi_k(z)|^2, \qquad \Omega(z) = 1 + \sum_{k=1}^{\infty} |\psi_k(z)|^2, \qquad \tilde{\omega}(z) = 1 + \sum_{k=1}^{\infty} |\chi_k(z)|^2.$$

The following inclusions hold (see [8, Theorem 11.8.2]):

$$\{\Omega_\mu(z) : \mu \in \mathcal{M}(\mathcal{R}_\infty)\} \subset \Delta_\infty(z) \subset \{\Omega_\mu(z) : \mu \in \mathcal{M}(\mathcal{L}_\infty)\}.$$

It follows from these inclusions that, if the moment problem on $\mathcal{R}_\infty$ is indeterminate, then $\Delta_\infty$ has to be a proper disk with positive radius, and hence the series $\sum_{k=1}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=1}^{\infty} |\psi_k(z)|^2$ converge. Furthermore, if the series $\sum_{k=1}^{\infty} |\varphi_k(z)|^2$ converges, then $\Delta_\infty$ is a proper disk, and hence the moment problem on $\mathcal{L}_\infty$ is indeterminate.

Now assume that the moment problem for $M$ on $\mathcal{R}_\infty$ is indeterminate. Then $\sum_{k=1}^{\infty} |\psi_k(z)|^2$ converges; hence also $\sum_{k=1}^{\infty} |\chi_k(z)|^2$ converges. Thus the associated moment problem for $\tilde{M}$ on $\tilde{\mathcal{L}}_\infty$ is indeterminate. *Because of the closely related recursion formulas, it is reasonable to expect that the moment problem for $\tilde{M}$ on $\tilde{\mathcal{R}}_\infty$ is indeterminate when the problem for $M$ on $\mathcal{R}_\infty$ is indeterminate. We have no proof of this, but we shall make this assumption in the proof of* (3.8) *and* Proposition 4.2 *for $F$ equal to $A$ or $C$*. However, when all the sets $\Gamma_\alpha$ are infinite, then $\mathcal{R}_\infty = \mathcal{L}_\infty$ and the moment problem on $\mathcal{L}_\infty$ and $\mathcal{R}_\infty$ coincide. In this case, $\tilde{\mathcal{R}}_\infty = \tilde{\mathcal{L}}_\infty = \mathcal{L}_\infty$, and thus the assumption above is automatically satisfied. Note also Remark 4.5, where the assumption is not needed. Thus our main result Theorem 4.4 does not depend on this assumption.

The theory of orthogonal rational functions with poles on the extended real line is equivalent to a theory of orthogonal rational functions with poles on the unit circle. See especially [8, Chapter 11] and [6].

For more details on the properties of orthogonal rational functions and rational moment problems that we have discussed so far, we refer to [2,6,7], [8, Chapter 11], [9].

The convergence results and the parameterization results below were obtained by Almendral in [2].

Assume that $\Delta_\infty(z)$ is a proper disk (the *limit circle case* in contrast to the *limit point case*). Then the functions $A_n, B_n, C_n, D_n$ converge locally uniformly in $\mathbb{C}_G$ to holomorphic functions $A, B, C, D$. We may then write

$$A(z) = H(z, x_0)\left[1 + \sum_{k=1}^{\infty} \psi_k(x_0)\psi_k(z)\right]$$

$$B(z) = H(z, x_0)\left[D(z, x_0) - \sum_{k=1}^{\infty} \psi_k(x_0)\varphi_k(z)\right]$$

$$C(z) = H(z, x_0)\left[D(z, x_0) + \sum_{k=1}^{\infty} \varphi_k(x_0)\psi_k(z)\right]$$

$$D(z) = H(z, x_0)\left[1 + \sum_{k=1}^{\infty} \varphi_k(x_0)\varphi_k(z)\right].$$

The collection $\{A, B, C, D\}$ is called a *Nevanlinna matrix* for the problem.

The functions $A, B, C, D$ appear in the following Nevanlinna-type parameterization for an indeterminate rational moment problem (see [2, Theorem 9]).

**Theorem 2.2.** *Assume that the moment problem on $\mathcal{R}_\infty$ is indeterminate, and consider the formula*

$$\Omega_\mu(z) = -\frac{A(z)f(z) - C(z)}{B(z)f(z) - D(z)}. \tag{2.3}$$

*Then*

(i) *For every Pick function $f$, there exists a $\mu \in \mathcal{M}(\mathcal{L}_\infty)$ such that (2.3) is satisfied.*
(ii) *For every $\mu \in \mathcal{M}(\mathcal{R}_\infty)$, there exists a Pick function $f$ such that (2.3) is satisfied.*

**Remark 2.3.** The correspondence between $\mu$ and $\Omega_\mu$ is one-to-one. When $\Gamma_\alpha$ is infinite for all $\alpha \in G$, we have

$$\{\Omega_\mu(z) : \mu \in \mathcal{M}(\mathcal{L}_\infty)\} = \Delta_\infty(z).$$

Hence in this situation (2.3) establishes a one-to-one correspondence between Pick functions and solutions of the moment problem (on $\mathcal{L}_\infty$ or $\mathcal{R}_\infty$).

## 3. A Riesz-type criterion

Let $\mu_1$ and $\mu_2$ be two distinct solutions of the moment problem on $\mathcal{R}_\infty$. The function $\Omega_{\mu_1}(z) - \Omega_{\mu_2}(z)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$; hence the zeros are isolated. It follows that there exist positive $\gamma, \gamma \neq 1$ such that $\Omega_{\mu_1}(\beta + \mathrm{i}\gamma) \neq \Omega_{\mu_2}(\beta + \mathrm{i}\gamma)$ for all $\beta \in \mathbb{R}$. Note that we then also have $S_{\mu_1}(\beta + \mathrm{i}\gamma) \neq S_{\mu_2}(\beta + \mathrm{i}\gamma)$ for all $\beta \in \mathbb{R}$. *Throughout the rest of this paper, we choose a fixed $\gamma$ with this property, and use the notation $\zeta_\beta = \beta + \mathrm{i}\gamma$. Without explicit mentioning, it will also be understood that the moment problem on $\mathcal{R}_\infty$ is indeterminate.*

In the following, *positive function* shall always mean *strictly positive function*.

**Proposition 3.1.** *Let $R$ be a function in $\mathcal{R}_\infty$ which is positive on $\mathbb{R}$. Then there exists a function $L \in \mathcal{L}_\infty$ such that*

$$\left|\frac{1}{x - \zeta_\beta} - L(x)\right| = \frac{\sqrt{R(x)}}{|x - \zeta_\beta|} \exp\left\{-\frac{\gamma}{2\pi}\int_{-\infty}^{\infty} \frac{\ln R(t)}{|t - \zeta_\beta|^2}\mathrm{d}t\right\}$$

*for all $x$ in $\mathbb{R}$ and $\zeta_\beta = \beta + \mathrm{i}\gamma, \gamma > 0$.*

**Proof.** By dividing out possible common factors in the numerator and denominator of $R$, we may write

$$R(z) = \frac{P(z)}{\left(1 - \frac{z}{\alpha_{k_1}}\right)^2 \cdots \left(1 - \frac{z}{\alpha_{k_p}}\right)^2},$$

where $P$ is a polynomial of degree $2n$, and $P(x)$ positive for $x \in \mathbb{R}$. The polynomial $P$ then has the form

$$P(z) = |A|^2 \left(1 - \frac{z}{\xi_1}\right) \cdots \left(1 - \frac{z}{\xi_n}\right) \left(1 - \frac{z}{\overline{\xi}_1}\right) \cdots \left(1 - \frac{z}{\overline{\xi}_n}\right)$$

for a suitable constant $A$ and $\xi_1, \ldots, \xi_n$ in the lower half-plane.

We define

$$Q(z) = \frac{A\left(1 - \frac{z}{\xi_1}\right) \cdots \left(1 - \frac{z}{\xi_n}\right)}{\left(1 - \frac{z}{\alpha_{k_1}}\right) \cdots \left(1 - \frac{z}{\alpha_{k_p}}\right)}.$$

Then clearly $|Q(x)|^2 = R(x)$ for $x \in \mathbb{R}$. We further define

$$L(z) = \frac{1 - \frac{Q(z)}{Q(\zeta_\beta)}}{z - \zeta_\beta}.$$

Note that $L \in \mathcal{L}_\infty$. We have

$$\frac{1}{z - \zeta_\beta} - L(z) = \frac{Q(z)}{(z - \zeta_\beta) Q(\zeta_\beta)};$$

hence, for $x \in \mathbb{R}$,

$$\left| \frac{1}{z - \zeta_\beta} - L(x) \right| = \frac{|Q(x)|}{|x - \zeta_\beta| |Q(\zeta_\beta)|} = \frac{\sqrt{R(x)}}{|x - \zeta_\beta| |Q(\zeta_\beta)|}. \tag{3.1}$$

Because $\ln |Q(z)|$ is harmonic in the upper half-plane, all the $\xi_i$ being in the lower half-plane, and because there are only a finite number of poles $\alpha_{k_j} \in \mathbb{R}$, we obtain by a general form of the Poisson formula that (see [15, Chapter 3])

$$\ln |Q(\zeta_\beta)| = \frac{\gamma}{\pi} \int_{-\infty}^{\infty} \frac{\ln |Q(t)|}{|t - \zeta_\beta|^2} \mathrm{d}t = \frac{\gamma}{2\pi} \int_{-\infty}^{\infty} \frac{\ln R(t)}{|t - \zeta_\beta|^2} \mathrm{d}t. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\left| \frac{1}{x - \zeta_\beta} - L(x) \right| = \frac{\sqrt{R(x)}}{|x - \zeta_\beta|} \exp\left\{ -\frac{\gamma}{2\pi} \int_{-\infty}^{\infty} \frac{\ln R(t) \mathrm{d}t}{|t - \zeta_\beta|^2} \right\},$$

which concludes the proof. □

**Corollary 3.2.** *For each non-negative integer $n$ and each $\alpha \in G$, there exists an $L_n \in \mathcal{L}_\infty$ such that, for $x \in \mathbb{R}$,*

$$\left| \frac{1}{x - \zeta_\beta} - L_n(x) \right| = \frac{\sqrt{1 + \omega_{\alpha,n}(x)}}{|x - \zeta_\beta|} \exp\left\{ -\frac{\gamma}{2\pi} \int_{-\infty}^{\infty} \frac{\ln[1 + \omega_{\alpha,n}(t)] \mathrm{d}t}{|t - \zeta_\beta|^2} \right\}.$$

**Proof.** The function $1 + \omega_{\alpha,n}(x)$ is the restriction to $\mathbb{R}$ of the function $1 + \sum_{k=1;\alpha_k \in \Gamma_\alpha}^{n-1} \varphi_k(z)^2$, which belongs to $\mathcal{R}_\infty$ and is positive on $\mathbb{R}$. Consequently, the result follows from Proposition 3.1. □

**Proposition 3.3.** *There exists a finite constant $K_1$ such that for every $R$ in $\mathcal{R}_\infty$ which is positive on $\mathbb{R}$ we have*

$$\exp\left\{\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln R(t)\mathrm{d}t}{|t-\zeta_\beta|^2}\right\} \leq K_1 \sup_{\mu\in\mathcal{M}(\mathcal{R}_\infty)}\left\{\int_{\mathbb{R}}\frac{\sqrt{R(x)}\mathrm{d}\mu(x)}{|x-\zeta_\beta|}\right\},$$

*where $K_1$ is independent of $R$.*

**Proof.** Recall that $S_{\mu_1}(\zeta_\beta) \neq S_{\mu_2}(\zeta_\beta)$, where $\mu_1, \mu_2$ are two different measures in $\mathcal{M}(\mathcal{R}_\infty)$; see the introduction to this section. Set $K_0 = S_{\mu_1}(\zeta) - S_{\mu_2}(\zeta) \neq 0$. Since $\int_{\mathbb{R}} L(x)\mathrm{d}\mu_1(x) = \int_{\mathbb{R}} L(x)\mathrm{d}\mu_2(x)$ for all $L \in \mathcal{L}_\infty$, we may write

$$K_0 = \int_{\mathbb{R}}\left(\frac{1}{x-\zeta_\beta} - L(x)\right)\mathrm{d}\mu_1(x) - \int_{\mathbb{R}}\left(\frac{1}{x-\zeta_\beta} - L(x)\right)\mathrm{d}\mu_2(x);$$

hence

$$|K_0| \leq \int_{\mathbb{R}}\left|\frac{1}{x-\zeta_\beta} - L(x)\right|\mathrm{d}\mu_1(x) + \int_{\mathbb{R}}\left|\frac{1}{x-\zeta_\beta} - L(x)\right|\mathrm{d}\mu_2(x),$$

and consequently

$$|K_0| \leq 2\sup\left\{\int_{\mathbb{R}}\left|\frac{1}{x-\zeta_\beta} - L(x)\right|\mathrm{d}\mu(x)\right\},$$

where the supremum is taken over all $\mu \in \mathcal{M}(\mathcal{R}_\infty)$.

Let $R$ be an arbitrary function in $\mathcal{R}_\infty$ which is strictly positive on $\mathbb{R}$. Then we conclude from Proposition 3.1 that

$$|K_0| \leq 2\exp\left\{-\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln R(t)\mathrm{d}t}{|t-\zeta_\beta|^2}\right\}\cdot\sup\left\{\int_{\mathbb{R}}\frac{\sqrt{R(x)}}{|x-\zeta_\beta|}\mathrm{d}\mu(x)\right\};$$

hence

$$\exp\left\{\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln R(t)\mathrm{d}t}{|t-\zeta_\beta|^2}\right\} \leq K_1\cdot\sup\left\{\int_{\mathbb{R}}\frac{\sqrt{R(t)}}{|x-\zeta_\beta|}\mathrm{d}\mu(x)\right\},$$

where $K_1 = 2/|K_0|$. □

**Corollary 3.4.** *There exists a constant $K_1$ independent of the index $n$ such that, for every $\alpha \in G$,*

$$\exp\left\{\frac{\gamma}{2\pi}\int_{-\infty}^{\infty}\frac{\ln[1+\omega_{\alpha,n}(t)]\mathrm{d}t}{|t-\zeta_\beta|^2}\right\} \leq K_1\cdot\sup\left\{\int_{\mathbb{R}}\frac{\sqrt{1+\omega_{\alpha,n}(t)}}{|x-\zeta_\beta|}\mathrm{d}\mu(x)\right\},$$

*where the supremum is taken over all $\mu \in \mathcal{M}(\mathcal{R}_\infty)$.*

**Proof.** The function $1+\omega_{\alpha,n}(x)$ is the restriction to $\mathbb{R}$ of the function $1+\sum_{k=1;\alpha_k\in\Gamma_\alpha}^{n-1}\varphi_k(z)^2$, which belongs to $\mathcal{R}_\infty$ and is positive on $\mathbb{R}$. Thus the conditions of Proposition 3.3 are satisfied, and so the result follows from this proposition. □

**Lemma 3.5.** *Let $\alpha \in G$, $\beta \in \mathbb{R}$. Then the following inequality holds for $\alpha_k = \alpha$, $\mu \in \mathcal{M}(\mathcal{R}_\infty)$:*

$$\left|\int_{\mathbb{R}}\frac{\left(1-\frac{x}{\alpha}\right)(x-\beta)\varphi_k(x)^2}{|x-\zeta_\beta|^2}\mathrm{d}\mu(x)\right|$$

$$\leq \frac{\left|1-\frac{\zeta_\beta}{\alpha}\right|}{|1-\gamma^2|}\left[|\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)|+|\Omega_\mu(\zeta_\beta)\varphi_k(\zeta_\beta)^2|\right]. \tag{3.3}$$

**Proof.** For any $\beta \in \mathbb{R}$, we may write

$$\frac{\left(1-\frac{x}{\alpha}\right)\varphi_k(x)^2}{x-\zeta_\beta} = \left(1-\frac{x}{\alpha}\right)\frac{\varphi_k(x)-\varphi_k(\zeta_\beta)}{x-\zeta_\beta}\varphi_k(x) - \frac{\zeta_\beta+\frac{1}{\alpha}}{1+\zeta_\beta^2}\varphi_k(\zeta_\beta)\varphi_k(x)$$
$$+\frac{1-\frac{\zeta_\beta}{\alpha}}{1+\zeta_\beta^2}\left[D(x,\zeta_\beta)\{\varphi_k(x)-\varphi_k(\zeta_\beta)\}\varphi_k(\zeta_\beta)+D(x,\zeta_\beta)\varphi_k(\zeta_\beta)^2\right].$$

We observe that $\left(1-\frac{x}{\alpha}\right)\frac{\varphi_k(x)-\varphi_k(\zeta_\beta)}{x-\zeta_\beta}$ belongs to $\mathcal{L}_{k-1}$. Thus the integral of the first term to the right vanishes by orthogonality. Also, the integral of the second term vanishes by orthogonality. We then get

$$\int_{\mathbb{R}} \frac{\left(1-\frac{x}{\alpha}\right)\varphi_k(x)^2}{x-\zeta_\beta}\mathrm{d}\mu(x) = \frac{1-\frac{\zeta_\beta}{\alpha}}{1+\zeta_\beta^2}\left[\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)+\varphi_k(\zeta_\beta)^2\Omega_\mu(\zeta_\beta)\right]. \tag{3.4}$$

Hence, by taking the real part of the equation, we get

$$\left|\int_{\mathbb{R}} \frac{\left(1-\frac{x}{\alpha}\right)(x-\beta)\varphi_k(x)^2}{|x-\zeta_\beta|^2}\mathrm{d}\mu(x)\right| \leq \frac{\left|1-\frac{\zeta_\beta}{\alpha}\right|}{|1+\zeta_\beta^2|}\left[|\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)|+|\varphi_k(\zeta_\beta)|^2|\Omega_\mu(\zeta_\beta)|\right].$$

We find that $|1+\zeta_\beta^2|^2 = (1+\beta^2-\gamma^2)^2+4\beta^2\gamma^2 \geq (1-\gamma^2)^2$, from which (3.3) now follows. □

The following result is obvious when choosing a positive $m(\beta) \leq$ the minimum of $|x-\zeta_\beta|^2/|x-\zeta_\alpha|^2$ on $\mathbb{R}$ and a finite $M(\beta) \geq$ its maximum on $\mathbb{R}$.

**Lemma 3.6.** *Let $\mu$ be a positive measure in $\mathbb{R}$ and let $f$ be a non-negative function in $L^1(\mathbb{R},\mu)$. Let $[a,b]$ be a bounded interval and let $\beta \in \mathbb{R}$. Then there exist positive numbers $m(\beta)$ and $M(\beta)$ such that*

$$m(\beta)\int_{\mathbb{R}}\frac{f(x)\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2} \leq \int_{\mathbb{R}}\frac{f(x)\mathrm{d}\mu(x)}{|x-\zeta_\alpha|^2} \leq M(\beta)\int_{\mathbb{R}}\frac{f(x)\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2}$$

*for all $\alpha \in [a,b]$.*

**Proposition 3.7.** *Assume that $G$ is bounded, $\alpha \in G, \beta \in \mathbb{R}$. Then there exists a constant $K_2(\alpha,\beta)$ independent of the index $n$ and the measure $\mu \in \mathcal{M}(\mathcal{R}_\infty)$ such that*

$$\int_{\mathbb{R}}\frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|}\mathrm{d}\mu(x) \leq K_2(\alpha,\beta). \tag{3.5}$$

**Proof.** We know that the series $\sum_{k=1}^{\infty}|\varphi_k(\zeta_\beta)|^2$ and $\sum_{k=1}^{\infty}|\psi_k(\zeta_\beta)|^2$ converge, and by the Schwarz inequality the series $\sum_{k=1}^{\infty}|\varphi_k(\zeta_\beta)\psi_k(\zeta_\beta)|$ also converges. Furthermore, $\Omega_\mu(\zeta_\beta) \in \Delta(\zeta_\beta)$, which implies that $\Omega_\mu(\zeta_\beta)$ is bounded independently of $\mu \in \mathcal{M}(\mathcal{R}_\infty)$.

It follows from Lemma 3.5 with $\alpha = \beta$, and since $\alpha \neq \infty$ because $G$ is bounded, that we can multiply out a factor $\alpha$ to obtain

$$\int_{\mathbb{R}} \frac{(x-\alpha)^2 \omega_{\alpha,n}(x)}{|x-\zeta_\alpha|^2} \mathrm{d}\mu(x) \leq \frac{\gamma}{|1-\gamma^2|} \sum_{\substack{k=1 \\ \alpha_k \in \Gamma_\alpha}}^{n-1} \left\{ |\varphi_k(\zeta_\alpha)\psi_k(\zeta_\alpha)| + |\Omega_\mu(\zeta_\alpha)\varphi_k(\zeta_\alpha)^2| \right\}.$$

Taking into account Lemma 3.6, we then get

$$\int_{\mathbb{R}} \frac{(x-\alpha)^2 \omega_{\alpha,n}(x)}{|x-\zeta_\beta|^2} \mathrm{d}\mu(x) \leq \frac{\gamma}{m(\beta)|1-\gamma^2|} \sum_{\substack{k=1 \\ \alpha_k \in \Gamma_\alpha}}^{n-1} \left\{ |\varphi_k(\zeta_\alpha)\psi_k(\zeta_\alpha)| + |\Omega_\mu(\zeta_\alpha)\varphi_k(\zeta_\alpha)^2| \right\}.$$

Thus there exists a constant $K_3(\alpha, \beta)$ independent of $n$ and $\mu \in \mathcal{M}(\mathcal{R}_\infty)$ such that

$$\int_{\mathbb{R}} \frac{(x-\alpha)^2 [1+\omega_{\alpha,n}(x)]}{|x-\zeta_\beta|^2} \mathrm{d}\mu(x) \leq K_3(\alpha, \beta). \tag{3.6}$$

We may write

$$\frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|} = \frac{|x-\alpha|\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|} \cdot \frac{1}{|x-\alpha|}.$$

Hence, by the Schwarz inequality, we get

$$\int_{\mathbb{R}} \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|} \mathrm{d}\mu(x) \leq \left[ \int_{\mathbb{R}} \frac{(x-\alpha)^2 [1+\omega_{\alpha,n}(x)]}{|x-\zeta_\beta|^2} \mathrm{d}\mu(x) \right]^{1/2} \cdot \left[ \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{(x-\alpha)^2} \right]^{1/2}. \tag{3.7}$$

The factor $1/(x-\alpha)^2$ belongs to $\mathcal{R}_\infty$; hence $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{(x-\alpha)^2}$ equals a finite constant $K_4$ (independent of $\mu$). Setting $K_2 = \sqrt{K_3(\alpha, \beta) K_4}$, we obtain (3.5) from (3.6) and (3.7). □

**Theorem 3.8** (*Riesz-Type Criterion*). *Assume that the moment problem on $\mathcal{R}_\infty$ is indeterminate. Assume that $G$ is finite and let $\beta \in \mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} \frac{\ln \omega(t) \mathrm{d}t}{|t-\zeta_\beta|^2} < \infty \quad \textit{and} \quad \int_{-\infty}^{\infty} \frac{\ln \Omega(t) \mathrm{d}t}{|t-\zeta_\beta|^2} < \infty. \tag{3.8}$$

**Proof.** According to Proposition 3.7, we have

$$\sup_{\mu \in \mathcal{M}(\mathcal{R}_\infty)} \int_{\mathbb{R}} \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|} \mathrm{d}\mu(x) \leq K_2(\alpha, \beta),$$

where $K_2(\alpha, \beta)$ is independent of $n$. Consequently, there is a constant $K_2(\beta)$ such that

$$\sum_{\alpha \in G} \sup_{\mu \in \mathcal{M}(\mathcal{R}_\infty)} \int_{\mathbb{R}} \frac{\sqrt{1+\omega_{\alpha,n}(x)}}{|x-\zeta_\beta|} \mathrm{d}\mu(x) \leq K_2(\beta).$$

It then follows from Corollary 3.4 that there is a constant $K(\beta)$ such that

$$\int_{-\infty}^{\infty} \sum_{\alpha \in G} \frac{\ln[1+\omega_{\alpha,n}(t)]}{|t-\zeta_\beta|^2} \mathrm{d}t \leq K(\beta) \quad \text{for all } n.$$

For any non-negative numbers $t_1, t_2, \ldots, t_N$, we have

$$\ln\left(1+\sum_{k=1}^{N} t_k\right) \le \sum_{k=1}^{N} \ln(1+t_k).$$

Consequently, we may conclude from the fact that $\omega_n(t) = 1 + \sum_{\alpha\in G} \omega_{\alpha,n}(t)$ that

$$\int_{-\infty}^{\infty} \frac{\ln \omega_n(t)}{|t-\zeta_\beta|^2} \mathrm{d}t \le \int_{-\infty}^{\infty} \frac{1}{|t-\zeta_\beta|^2} \left\{\sum_{\alpha\in G} \ln[1+\omega_{\alpha,n}(t)]\right\} \mathrm{d}t \le K(\beta)$$

for all $n$, from which the first inequality in (3.8) follows.

Similarly, since $\{\chi_n\}$ are the orthonormal functions associated with the indeterminate moment problem on $\tilde{\mathcal{R}}_\infty$, we find

$$\int_{-\infty}^{\infty} \frac{\ln \tilde{\omega}_n(t)}{|t-\zeta_\beta|^2} \mathrm{d}t < \infty.$$

Then, from (2.1) and (2.2), we infer that the second inequality in (3.8) is also satisfied. $\square$

**Remark 3.9.** It was assumed in the previous proofs that $\alpha \neq \infty$ ($G$ was bounded). However, by considering the imaginary part in (3.4) when $G = \{\infty\}$, this means that we have to replace in the subsequent formula $(1 - x/\alpha)(x - \beta)$ by $\gamma$, and we can bound it by a constant depending only on $\beta$. So we can add the terms with $\alpha = \infty \in G$ and still get $\left|\int_{\mathbb{R}} \frac{\omega_n(x)\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2}\right| \le \frac{K_3}{\gamma}$, and hence by the Schwarz inequality

$$\int_{\mathbb{R}} \frac{\sqrt{\omega_n(x)}\mathrm{d}\mu(x)}{|x-\zeta_\beta|} \le \left[\int_{\mathbb{R}} \frac{\omega_n(x)\mathrm{d}\mu(x)}{|x-\zeta_\beta|^2}\right]^{1/2} \le \sqrt{\frac{K_3}{\gamma}}.$$

It follows from Corollary 3.4 that in this case $\int_{-\infty}^{\infty} \frac{\ln \omega(t)\mathrm{d}t}{|t-\zeta_\beta|^2} < \infty$.

## 4. Growth estimates in the finite case

We continue to assume that the moment problem on $\mathcal{R}_\infty$ is indeterminate. Let $\alpha$ be a fixed point in $G$. For the sake of simplicity, we formulate the results and carry out the arguments only when $\alpha \neq \infty$. By adapting the arguments given in this section, estimates in appropriate form can also be proved when $\alpha = \infty$. In the following, $\beta$ shall denote an arbitrary point in $G$.

Recall that $D(t,z) = (1+tz)/(t-z)$ and $H(t,z) = 1/t - 1/z$, and $x_0 \in \mathbb{R} \setminus [\hat{G} \cup \{0\}]$. We set $m(t) = \max\{1, |D(t,x_0)|\}$, $p(t) = \max\{1, |H(t,x_0)|\}$, $q = \max\{[\sum_{k=1}^{\infty} \varphi_k(x_0)^2]^{1/2}, [\sum_{k=1}^{\infty} \psi_k(x_0)^2]^{1/2}\}$, and define

$$\Phi(t) = \ln\left[p(t)\left\{m(t) + q\sqrt{\omega(t)}\right\}\right], \qquad \Psi(t) = \ln\left[p(t)\left\{m(t) + q\sqrt{\Omega(t)}\right\}\right].$$

It follows from Theorem 3.8 (the Riesz-type criterion) that

$$\int_{-\infty}^{\infty} \frac{\Phi(t)\mathrm{d}t}{|t-\zeta_\beta|^2} < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\Psi(t)\mathrm{d}t}{|t-\zeta_\beta|^2} < \infty \quad \text{for any } \beta \in \mathbb{R}. \tag{4.1}$$

Note that $\Phi(t) \ge 0$ and $\Psi(t) \ge 0$ for all $t \in \mathbb{R}$.

Let $\eta \in (0, \pi/2)$. We introduce the notation

$$\Delta(\alpha,\eta) = \{z \in \mathbb{C} : \eta \le |\arg(z-\alpha)| \le \pi - \eta\}.$$

As usual, we set $z = x + y\mathrm{i}$ and we recall the notation $\zeta_\alpha = \alpha + \mathrm{i}\gamma$ as defined at the beginning of the previous section.

**Lemma 4.1.** *Assume that $f$ is a non-negative function on $\mathbb{R}$ satisfying $\int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{|t-\zeta_\alpha|^2} < \infty$ for some $\alpha \in \mathbb{R}$. Then for every $\varepsilon > 0$ there exists a disk $U_\alpha$ with center at $\alpha$ such that*

$$\frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{f(t)\mathrm{d}t}{|t-z|^2} < \frac{\varepsilon}{|z-\alpha|} \tag{4.2}$$

*for $z \in U_\alpha \cap \Delta(\alpha, \eta)$.*

**Proof.** Since $|t-\zeta_\alpha|^2 = (t-\alpha)^2 + \gamma^2 \le 2\gamma^2$ for $|t-\alpha| \le \gamma$, we have

$$\frac{1}{2\gamma^2} \int_{|t-\alpha|\le\gamma} f(t)\mathrm{d}t \le \int_{|t-\alpha|\le\gamma} \frac{f(t)\mathrm{d}t}{|t-\zeta_\alpha|^2} < \infty,$$

and thus $\int_{|t-\alpha|\le\gamma} f(t)\mathrm{d}t < \infty$. Furthermore, $\frac{y^2 f(t)}{|t-z|^2} \le f(t)$ a.e., so that also $\int_{|t-\alpha|\le\gamma} \frac{f(t)\mathrm{d}t}{|t-z|^2} < \infty$. It follows by Lebesgue's dominated convergence theorem that

$$\frac{y^2}{\pi} \int_{|t-\alpha|\le\gamma} \frac{f(t)\mathrm{d}t}{|t-z|^2} \underset{y\to 0}{\longrightarrow} 0; \quad \text{hence } \frac{y^2}{\pi} \int_{|t-\alpha|\le\gamma} \frac{f(t)\mathrm{d}t}{|t-z|^2} < \frac{\varepsilon}{2} \sin\eta \tag{4.3}$$

for $|y|$ sufficiently small.

For $z \in \Delta(\alpha, \eta)$, we have $|t-z|^2 \ge |t-\alpha|^2 \sin^2\eta$. For $|t-x| \ge \gamma$, this implies that $|t-x|^2 \ge \frac{1}{2}|t-\zeta_\alpha|^2 \sin^2\eta$. Hence,

$$\frac{y^2}{\pi} \int_{|t-\alpha|\ge\gamma} \frac{f(t)\mathrm{d}t}{|t-z|^2} \le \frac{2y^2}{\pi \sin^2\eta} \int_{|t-\alpha|\ge\gamma} \frac{f(t)\mathrm{d}t}{|t-\zeta_\alpha|^2} < \infty.$$

So $\int_{|t-\alpha|\ge\gamma} \frac{f(t)\mathrm{d}t}{|t-z|^2} < \infty$, and we find that

$$\frac{y^2}{\pi} \int_{|t-\alpha|\ge\gamma} \frac{f(t)\mathrm{d}t}{|t-z|^2} \underset{y\to 0}{\longrightarrow} 0; \quad \text{hence } \frac{y^2}{\pi} \int_{|t-\alpha|\ge\gamma} \frac{f(t)\mathrm{d}t}{|t-z|^2} < \frac{\varepsilon}{2} \sin\eta \tag{4.4}$$

for $|y|$ sufficiently small.

We have $|z-\alpha| \sin\eta < |y|$ when $z \in \Delta(\alpha, \eta)$, and so (4.2) follows from (4.3) and (4.4). □

Our goal is to estimate the growth of the functions $A$, $B$, $C$ and $D$ in a disk $V_\alpha$ centered at a point $\alpha \in G$. We shall first give a bound in $V_\alpha \cap \Delta(\alpha, \eta)$ in Proposition 4.2, and in Proposition 4.3 we give the bound for the remaining part of $V_\alpha$, which then immediately result in Theorem 4.4.

**Proposition 4.2.** *Assume that $G$ is finite, and let $\alpha \in G$. Let $V_\alpha$ be a disk with center at $\alpha$ and let $F$ denote any of the functions $A, B, C, D$. Then there exists for every $\varepsilon > 0$ a constant $M_1(\varepsilon, \eta)$ such that*

$$|F(z)| \le M_1(\varepsilon, \eta) \exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\} \tag{4.5}$$

*for $z \in V_\alpha \cap \Delta(\alpha, \eta)$.*

**Proof.** Let $H_n$ denote any of the functions $B_n, D_n$. It follows by the Schwarz inequality that $|H_n(t)| \le p(t)[m(t) + q\sqrt{\omega(t)}]$; hence $\ln|H_n(t)| \le \Phi(t)$ for $t \in \mathbb{R}$.

Recall that all the zeros and poles of $H_n$ are real.

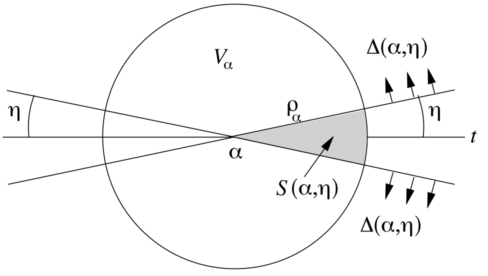


Fig. 1. Elements appearing in the proof of Proposition 4.3.

From Poisson's formula, applied to the rational function $H_n(t)$ with all poles on $\mathbb{R}$, we find that

$$\ln|H_n(z)| = \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\ln|H_n(t)|\mathrm{d}t}{|t-z|^2} \le \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\Phi(t)\mathrm{d}t}{|t-z|^2} \quad \text{for } z \notin \mathbb{R}.$$

It now follows from (4.1) and Lemma 4.1 that $\ln|H_n(z)| \le \frac{\varepsilon}{|z-\alpha|}$ for $z \in U_\alpha \cap \Delta(\alpha,\eta)$, where $U_\alpha$ is sufficiently small. In $(V_\alpha \setminus U_\alpha) \cap \Delta(\alpha,\eta)$, we have $\ln|H_n(z)| \le M_1^*(\varepsilon,\eta)$, where the constant $M_1^*(\varepsilon,\eta)$ is independent of $n$, since $H_n$ is uniformly convergent in $(V_\alpha \setminus U_\alpha) \cap \Delta(\alpha,\eta)$. Thus

$$\ln|H_n(z)| \le M_1^*(\varepsilon,\eta) + \frac{\varepsilon}{|z-\alpha|}$$

in $V_\alpha \cap \Delta(\varepsilon,\eta)$ for all $n$.

Similarly, let $G_n$ denote any of the functions $A_n, C_n$. Then $\ln|G_n(t)| \le \Psi(t)$ for all $t \in \mathbb{R}$, and all the zeros and poles of $G_n$ are real. It follows from (4.1) by the same kind of reasoning as above that there exists a constant $M_1^{**}(\varepsilon,\eta)$ such that

$$\ln|G_n(z)| \le M_1^{**}(\varepsilon,\eta) + \frac{\varepsilon}{|z-\alpha|} \quad \text{for } z \in V_\alpha \cap \Delta(\alpha,\eta).$$

Setting $M_1(\varepsilon,\eta) = \max\{\exp[M_1^*(\varepsilon,\eta)], \exp[M_1^{**}(\varepsilon,\eta)]\}$, we obtain (4.5). $\square$

**Proposition 4.3.** *Assume that $G$ is finite, $\infty \notin G$. Let $\alpha \in G$ and let $V_\alpha$ be a disk with center at $\alpha$ containing no other point in $G$. Then for every $\varepsilon > 0$ there exists a constant $M_2(\varepsilon,\eta)$ such that*

$$|F(z)| \le M_2(\varepsilon,\eta)\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}$$

*for $z \in V_\alpha \cap [\mathbb{C} \setminus \Delta(\alpha,\eta)]$, where $F$ is any of the functions $A, B, C, D$ (see Fig. 1).*

**Proof.** Let $\rho_\alpha$ denote the radius of $V_\alpha$ and let $S(\alpha,\eta)$ denote the sector of $V_\alpha \cap [\mathbb{C} \setminus \Delta(\alpha,\eta)]$ lying to the right of $\alpha$. According to Proposition 4.2, there is for every $\varepsilon > 0$ a constant $M_1(\varepsilon\cos\eta,\eta)$ such that $|F(z)| \le M_1(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon\cos\eta}{|z-\alpha|}\right\}$ for $z \in V_\alpha \cap \Delta(\alpha,\eta)$, and hence a constant $M_2^*(\varepsilon\cos\eta,\eta)$ such that

$$|F_n(z)| \le M_2^*(\varepsilon\cos\eta,\eta)\exp\left\{\frac{\varepsilon\cos\eta}{|z-\alpha|}\right\} \tag{4.6}$$

for $z \in V_\alpha \cap \Delta(\alpha,\eta)$, where $F_n$ is any of the functions $A_n, B_n, C_n, D_n$.

We now consider $z$ in the closure $\overline{S}(\alpha, \eta)$ of the sector $S(\alpha, \eta)$. The function

$$Q_n(z) = F_n(z) \exp\left\{-\frac{\varepsilon}{z-\alpha}\right\}$$

is holomorphic in $\overline{S}(\alpha, \eta) \setminus \{\alpha\}$. We have

$$|Q_n(z)| = |F_n(z)| \exp\left\{-\varepsilon \operatorname{Re} \frac{1}{z-\alpha}\right\}; \tag{4.7}$$

hence, by (4.6), and since $z \in \partial S(\alpha, \eta) \setminus \{\alpha\}$,

$$|Q_n(z)| \le M_2^*(\varepsilon \cos\eta, \eta) \exp\left\{\frac{\varepsilon \cos\eta}{|z-\alpha|}\right\} \cdot \exp\left\{-\varepsilon \frac{x-\alpha}{|z-\alpha|^2}\right\}. \tag{4.8}$$

Let $z$ be a point on one of the line segments of the boundary $\partial S(\alpha, \eta)$. Then $x - \alpha = |z-\alpha| \cos\eta$; hence

$$|Q_n(z)| \le M_2^*(\varepsilon \cos\eta, \eta). \tag{4.9}$$

Next let $z$ be a point on the circular arc of $\partial S(\alpha, \eta)$. Then, we have

$$|Q_n(z)| \le M_2^*(\varepsilon \cos\eta, \eta) \exp\left\{\frac{\varepsilon \cos\eta}{\rho_\alpha}\right\} \exp\left\{-\frac{\varepsilon(x-\alpha)}{\rho_\alpha^2}\right\};$$

hence, since $x > \alpha$, and thus the last exponential being less than 1,

$$|Q_n(z)| \le M_2^*(\varepsilon \cos\eta, \eta) \exp\left\{\frac{\varepsilon \cos\eta}{\rho_\alpha}\right\}.$$

Thus there is a constant $M_3^*(\varepsilon \cos\eta, \eta)$ such that

$$|Q_n(z)| \le M_3^*(\varepsilon \cos\eta, \eta) \quad \text{for } z \in \partial S(\alpha, \eta) \setminus \{\alpha\}.$$

Recall that $F_n$ is a rational function. Therefore there exists for every $\varepsilon > 0$ a constant $k_n(\varepsilon)$ such that $|F_n(z)| \le k_n(\varepsilon) \exp\left\{\frac{\varepsilon \cos\eta}{|z-\alpha|}\right\}$ for $z \in V_\alpha$. For $z \in S(\alpha, \eta) \setminus \{\alpha\}$, we have $|x - \alpha| \ge |z-\alpha| \cos\eta$. Hence

$$|Q_n(z)| \le k_n(\varepsilon) \exp\left\{\frac{\varepsilon \cos\eta}{|z-\alpha|}\right\} \exp\left\{-\varepsilon \frac{x-\alpha}{|z-\alpha|^2}\right\} \le k_n(\varepsilon)$$

for $z \in \overline{S}(\alpha, \eta) \setminus \{\alpha\}$. It follows that

$$\limsup_{\substack{z \to \alpha \\ z \in S(\alpha,\eta)\setminus\{\alpha\}}} |Q_n(z)| \le k_n(\varepsilon) < \infty.$$

Then, according to a version of the maximum principle (see for example [16, Part II, p. 208]), we have

$$|Q_n(z)| \le M_3^*(\varepsilon \cos\eta, \eta) \quad \text{for } z \in S(\alpha, \eta),$$

and hence, according to (4.7),

$$|F_n(z)| \le M_3^*(\varepsilon \cos\eta, \eta) \exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}, \quad \text{for } z \in S(\alpha, \eta). \tag{4.10}$$

In the same way, we find an estimate

$$|F_n(z)| \le M_3^{**}(\varepsilon\cos\eta, \eta)\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}, \tag{4.11}$$

for $z$ in the sector of $V_\alpha \cap [\mathbb{C} \setminus \Delta(\alpha, \eta)]$ to the left of $\alpha$.

Letting $n$ tend to infinity in (4.10)–(4.11) and combining the resulting inequalities, the proof is completed. □

**Theorem 4.4.** *Assume that $G$ is finite, $\infty \notin G$. Let $\alpha \in G$ and let $V_\alpha$ be a disk with center at $\alpha$ containing no other point of $G$. Then for every $\varepsilon > 0$ there exists a constant $M(\varepsilon)$ such that*

$$|F(z)| \le M(\varepsilon)\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}$$

*for all $z \in V_\alpha$, where $F$ is any of the functions $A, B, C, D$.*

**Proof.** Choose a fixed $\eta = \eta_0$, and define $M(\varepsilon) = \max\{M_1(\varepsilon, \eta_0), M_2(\varepsilon, \eta_0)\}$. The result then follows from Propositions 4.2 and 4.3. □

**Remark 4.5.** For fixed $t \in \hat{\mathbb{R}}$, the rational function

$$T_n(z, t) = -\frac{A_n(z)t - C_n(z)}{B_n(z)t - D_n(z)}$$

has a partial fraction decomposition of the form

$$T_n(z, t) = \sum_{k=1}^{n} \lambda_{n,k}(t)\frac{1 + \xi_{n,k}(t)z}{\xi_{n,k}(t) - z},$$

with $\xi_{n,k} \in \mathbb{R}$ and $\lambda_{n,k} > 0$ for $k = 1, \dots, n$, and $\sum_{k=1}^{n} \lambda_{n,k}(t) = 1$. (See [2, Sections 10–11], [8, Section 11.10].) Since $|\xi_{n,k}(t) - z| \ge |y| \ge |z-\alpha|\sin\eta$ for $z \in \Delta(\alpha, \eta)$, we find that $|T_n(z,t)| \le \frac{1+|z|^2}{|z-\alpha|} + |z|$. In particular,

$$\left|\frac{A_n(z)}{B_n(z)}\right| \le \frac{1+|z|^2}{|z-\alpha|} + |z|, \qquad \left|\frac{C_n(z)}{D_n(z)}\right| \le \frac{1+|z|^2}{|z-\alpha|} + |z|,$$

for $z \in \Delta(\alpha, \eta)$. Since $\frac{1}{|z-\alpha|} \le m\exp\left\{\frac{\varepsilon'}{|z-\alpha|}\right\}$ for arbitrary $\varepsilon' > 0$ and suitable $m$, we conclude that, if $B$ and $D$ satisfy an estimate of the form $|F(z)| \le M\exp\left\{\frac{\varepsilon}{|z-\alpha|}\right\}$ for $z \in \Delta(\alpha, \eta)$ for some $M$, then also $A$ and $C$ do. Hence the result of Proposition 4.2 can be obtained without making use of the Riesz-type criterion $\int_{-\infty}^{\infty} \frac{\ln \Omega(t)}{|t-\zeta_\beta|^2}\mathrm{d}t < \infty$ (see (3.8)). The result in Theorem 4.4 can thus be established without use of this criterion.

**Remark 4.6.** When $G$ consists of only the point $\infty$ (i.e. when $\alpha_k = \infty$ for all $k$), the functions $F_n$ are polynomials with all zeros in $\mathbb{R}$. Hence $|F_n(z)|$ is an increasing function of $y$, and an estimate of the form $|F_n(z)| \le M(\varepsilon, \eta)\exp\{\varepsilon|z|\}$ in $\{z \in \mathbb{C} : \eta \le |\arg z| \le \pi - \eta\}$ can easily be extended to an estimate of the same kind in the whole plane. See e.g. [1, Chapter 2]. An argument of this kind is not possible in the general case.

**Remark 4.7.** If $\alpha$ is an isolated point in $G$ and there is only a finite number $m$ of elements $\alpha_k$ in some $\Gamma_\alpha$, then $F$ has a pole of order $m$ at $\alpha$. Thus in a neighborhood $V_\alpha$ we have in this case the stronger estimate $|F(z)| \le \tilde{M}(\varepsilon)|z-\alpha|^{-m}$.

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